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Vertex-Transitive Graphs and Accessibility

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We call an infinite graph G accessible if there exists a natural number k such that any two ends of G can be separated by k edges. C. T. C. Wall's accessibility conjecture for finitely generated groups has a simple and attractive graph version: Every locally finite Cayley graph is accessible. Wall's conjecture has recently been disproved by M. J. Dunwoody. In this paper we show that all locally finite, 2-transitive graphs and all 1-transitive graphs of prime degree are accessible. We prove that every locally finite, vertex-transitive graph with at least one thick end has a thick end with a 2-way infinite geodesic, while no thin end has a 2-way infinite geodesic. Furthermore, those ends in a locally finite, accessible vertex-transitive graph which have a 2-way infinite geodesic are precisely the thick ends. In addition, there are only finitely many non-isomorphic thick ends. We obtain these and other results from a precise description of the end structure of every locally finite, accessible, vertex-transitive graph. We also investigate the ends in inaccessible vertex-transitive graphs. © 1993 Academic Press, Inc.

1. INTRODUCTION

J. Stallings [15] proved that a finitely generated group Γ with more than one end splits as a non-trivial free product with finite amalgamation or as an HNN-extension over a finite subgroup. If one of the factors has more than one end, this splitting continues. If the splitting cannot continue indefinitely, the group is *accessible*. C. T. C. Wall [17] conjectured that

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every finitely generated group is accessible. Dunwoody [5] verified this for finitely presented groups. Recently, he disproved Wall's conjecture [6]. Dicks and Dunwoody [1] reformulated Wall's conjecture as a problem concerning a certain Boolean ring associated with some Cayley graph of the group. In Section 7 we translate that result into Theorem 1.1 below.

DEFINITION. A graph G is *accessible* if there exists a natural number k such that, for any two ends ω_1 and ω_2 of G, there is a set E of at most k edges in G such that E separates ω_1 and ω_2. (Ends of graphs and separation of ends are defined in Section 4.)

THEOREM 1.1. *A finitely generated group is accessible if and only if some (and hence each) of its Cayley graphs are accessible.*

We prove that every 2-transitive graph and every 1-transitive r-regular graph (r prime) either has only one end or is a tree. In particular, every such graph is accessible. (k-transitive graphs are defined in Section 3.)

We present other equivalent versions of accessibility of locally finite vertex-transitive graphs. For example, it is sufficient to prove that, for each end of the graph, there exists some natural number k (depending on the end) such that the end can be separated from any other end by k edges. Another equivalent version says that there exists a number d such that all thin ends have diameter $\leqslant d$. This settles [19, Problem 2]. (An end is *thin* if it does not contain an infinite collection of pairwise disjoint infinite paths. Its diameter is defined in Section 4.)

We apply the powerful theory of non-crossing cuts and their associated structure trees developed by Dicks and Dunwoody [1] to give a fairly precise description of the end structure of each connected, locally finite, accessible, vertex-transitive graph. We conclude that any such graph has only finitely many non-isomorphic thick ends in the sense that the automorphism group of the graph acts on the thick ends with only finitely many orbits. We prove that a connected, locally finite, vertex-transitive graph with no thick end is accessible and thereby obtain, from the above structure result, the result of Woess [19] that every such graph is "tree-like."

Our results indicate that locally finite inaccessible vertex-transitive graphs must have "complicated" thick ends. It turns out that 2-way infinite geodesics are useful for analysing thick ends. Geodesics in vertex-transitive graphs were studied by Watkins [18]. We say that a 2-way infinite path P in a graph G *belongs to the end* ω in G if both ends of P are in ω. We show that, in a connected, locally finite, vertex-transitive graph G, no thin end has a 2-way infinite geodesic. We conjecture that every thick end has such a path and we verify this for accessible graphs. In the non-accessible case we show that some thick end has a 2-way infinite geodesic. If ω is a thick end in a connected, locally finite, vertex-transitive graph G and $G(\omega)$ is the union of the 2-way infinite geodesics in G belonging to ω, then we show that every end of $G(\omega)$ belongs to ω. In the accessible case we show that G has a connected, almost transitive subgraph $G'(\omega)$ such that $G'(\omega)$ contains $G(\omega)$, and $G'(\omega)$ has only one end (which in G is ω).

2. Terminology and Preliminaries

A *graph* G consists of a set $V(G)$ of *vertices* and a set $E(G)$ of unordered pairs xy of vertices called *edges*. We say that the edge xy *joins* x and y and that it is *incident* with x and y. The number of edges incident with x is the *degree* of x and is denoted $d_G(x)$. A graph is *r-regular* if all vertices have degree r. A graph is *locally finite* if all degrees are finite. A *path* ... $v_1 v_2 v_3$ ... is a graph consisting of distict vertices ..., $v_1, v_2, v_3, \ldots$ and all edges between consecutive vertices. A path may be finite or 1-way infinite or 2-way infinite. If it is finite, the number of edges in it is the *length* of the path. A graph is *connected* if any two vertices are joined by a path. A *component* in a disconnected graph is a maximal connected subgraph. The length of a shortest path from x to y in a connected graph G is called the *distance* between x and y and is denoted $\mathrm{dist}_G(x, y)$. More generally, if $X, Y \subseteq V(G) \cup E(G)$, then $\mathrm{dist}_G(X, Y)$ is the length of a shortest path from an (end)vertex of X to an (end)vertex of Y. A path is a *geodesic* if each finite subpath is a shortest path between its endvertices. A *cycle* is a finite connected 2-regular graph. A *tree* is a connected graph with no cycle.

If $S \subseteq V(G) \cup E(G)$, then $G - S$ is the subgraph obtained from G by deleting S and all edges incident with vertices of S.

If $S \subseteq V(G)$, then $G(S)$, the subgraph of G *induced* by S, is the graph $G - (V(G) \setminus S)$. If $F \subseteq E(G)$, then $G[F]$ is the graph consisting of F and all endvertices of F. We write $G - x$ instead of $G - \{x\}$ when $x \in V(G) \cup E(G)$. If G is connected and $G - x$ is disconnected, then x is a *cutvertex* (if $x \in V(G)$) or a *bridge* (if $x \in E(G)$).

An *automorphism* of a graph G is a bijection φ: $V(G) \to V(G)$ taking neighbours to neighbours and non-neighbours to non-neighbours. G is *vertex-transitive* (respectively *almost transitive*) if $V(G)$ has only one (respectively finitely many) orbit(s) under the automorphism group $\mathrm{Aut}(G)$. An *edge-transitive* graph is defined analogously. If $E \subseteq E(G)$ and $\varphi \in \mathrm{Aut}(G)$, then $\varphi(E)$ is defined as all $\varphi(x)\,\varphi(y)$, where $xy \in E$. Two vertex sets or edge sets or subgraphs in G are called *G-equivalent* if there is an automorphism of G taking one onto the other.

If the vertex set of a connected graph G is partitioned into sets A and B, then the set F of edges between A and B is called a *cut* or a *k-cut* if $|F| = k$.

A and B are called the *sides* of the cut F. We say that F *separates* any subgraph of $G(A)$ from any subgraph of $G(B)$. If both $G(A)$ and $G(B)$ are connected we say that F is a *tight cut*. If F and F' are cuts with sides A, B and A', B' respectively, then F and F' *cross* (and are called *crossing cuts*) if all four sets $A \cap A'$, $A \cap B'$, $B \cap A'$, and $B \cap B'$ are non-empty. Dunwoody [4, 1] proved the following:

THEOREM 2.1. *Every infinite connected graph which has a finite cut with infinite sides also has a finite tight cut F such that both sides of F are infinite and such that F crosses no $\varphi(F)$, where $\varphi \in \mathrm{Aut}(G)$.*

Note that in [4] it is not stated that the cut F can be chosen to be tight. However, this is part of the substantial extension of [4] given in [1], see in particular [1, 2.1, 2.20, and 2.21]. A finite tight cut F satisfying Theorem 2.1 will be called a *D-cut*.

This remarkable result has significant applications in combinatorial group theory. In the next section we obtain further applications.

3. Accessibility of k-Transitive Graphs

A connected locally finite graph is said to have *more than one end* if it has a finite cut with infinite sides; otherwise we say that it has *one end*. For a more precise definition of ends, see Section 4.

A graph G is said to be *k-transitive* (where k is a natural number) if G has no cycle of length $\leqslant k$ and, for any two paths $x_0 x_1 \cdots x_m$ and $y_0 y_1 \cdots y_m$ $(0 \leqslant m \leqslant k)$, there is an automorphism φ of G such that $\varphi(x_i) = y_i$ for $i = 0, 1, ..., m$. 1-transitive implies edge-transitive, but not conversely.

Tutte [16] proved that a finite 3-regular graph cannot be k-transitive for $k > 5$. It is a well-known unsolved problem if there exist infinite, connected, locally finite, k-transitive graphs, (other than the regular trees), for arbitrarily large k. We show that, for $k \geqslant 2$ they must be one-ended. The same holds if $k = 1$ and the vertex degrees are a prime. Seifter [14] proved that a connected, vertex-transitive graph of polynomial growth and with vertex degrees at least 3 cannot be 8-transitive. Below, two edges are *independent* if they have no common endvertex.

LEMMA 3.1. *If G is a connected, locally finite, edge-transitive graph with more than one end, then no D-cut of cardinality $\geqslant 2$ of G consists of pairwise independent edges.*

Proof. Suppose (reductio ad absurdum) that F is a D-cut in G consisting of two or more independent edges. Let A_1, A_2 be the two sides of F. Let

P be a shortest path in G-F connecting two endvertices of distinct edges in F, say $P \subseteq G(A_2)$. Let φ be an automorphism of G such that $\varphi(F)$ intersects $E(P)$. Let B_1, B_2 be the sides of $\varphi(F)$. As $B_1 \cap A_2 \neq \emptyset$ and $B_2 \cap A_2 \neq \emptyset$ (because P is not contained in either of B_1, B_2) and F is a D-cut, we conclude that $A_1 \subseteq B_1$ or $A_1 \subseteq B_2$, say $A_1 \subseteq B_1$. If P has at least two edges in common with $\varphi(F)$, then either $G(B_1)$ or $G(B_2)$ contains a proper subpath P' of P between these edges. But then $\varphi^{-1}(P')$ contradicts the minimality of P. So P has only one edge e in common with $\varphi(F)$. Hence P has one endvertex x_1 which belongs to B_1 and an endvertex x_2 in B_2. Now the edge in F from x_2 to A_1 is also in $\varphi(F)$, but it is not an edge of P. Let P' be the proper subpath of P between x_2 and e. Since $P' \subseteq G(B_2)$, $\varphi^{-1}(P')$ contradicts the minimality of P. ▮

THEOREM 3.2. *If G is a connected, locally finite, 2-transitive graph with more than one end, then G is a regular tree.*

Proof. By Theorem 2.1, G has a D-cut F. Let x be a vertex of largest degree, m say, in $G[F]$. If $|F| = 1$, then some and hence each edge of G is a bridge. Hence G is a tree. So assume $|F| \geqslant 2$. By Lemma 3.1, $m \geqslant 2$. Let x_1, x_2 be two neighbours of x in $G[F]$. As F is a tight cut, x is also incident with an edge xy not in F. Let A_1, A_2 be the two sides of F, so that $x, y \in A_1$, $x_1, x_2 \in A_2$. Now let φ be an automorphism of G such that the path $x_1 x x_2$ is mapped onto $x_1 x y$. Let B_1, B_2 be the two sides of $\varphi(F)$, so that $x \in B_1$ and $y, x_1 \in B_2$. As F is a D-cut. A_2 is contained in one of B_1, B_2. Since $x_1 \in B_2$, $A_2 \subseteq B_2$. But then $\varphi(F)$ contains all m edges from x to A_2 as well as the edge xy. Hence F contains $m+1$ edges incident with the same vertex. This contradicts the maximality of m. ▮

THEOREM 3.3. *Let G be a connected, locally finite, 1-transitive, r-regular graph, where r is a prime. If G has more than one end, then G is an r-regular tree.*

Proof. Let x, m, A_1, A_2 be as in the proof of Theorem 3.2. We shall prove that $m = 1$ in which case G is a tree. So suppose (reductio ad absurdum) that $m \geqslant 2$. Let x_1, x_2, y be as in the proof of Theorem 3.2 and let φ be an automorphism such that $\varphi(x) = x$, $\varphi(x_1) = y$. Let B_1 be the side of $\varphi(F)$ containing x, and let B_2 be the other side of $\varphi(F)$. Then $y \in B_2$. If B_2 contains some (and hence each) vertex of A_2, then we obtain a contradiction to the maximality of m as in the proof of Theorem 3.2. So $A_2 \subseteq B_1$. Moreover, $\varphi(F)$ has m edges from x to B_2 (because φ is an isomorphism). Let I denote the set of m edges from x to A_2.

We have proved the following: If φ is an automorphism of G such that $\varphi(x) = x$, then either $\varphi(I) = I$ or $\varphi(I) \cap I = \emptyset$. It follows that the edge sets of the form $\varphi(I)$ (where $\varphi(x) = x$) partition the set of r edges incident with x into, say q, pairwise disjoint sets where $q \geqslant 2$. But then $r = qm$ contradicting the assumption that r is prime. ∎

Theorem 3.3 does not extend to the case where r is not a prime. For if $r = mq$ where $m, q \geqslant 2$, then we obtain a 1-transitive graph from the m-regular tree by replacing each vertex by a set of q vertices and joining two q-vertex-sets completely if they correspond to neighbours in the tree. It would be interesting to characterize completely the connected, locally finite, 1-transitive graphs with more than one end.

Let us say that a graph G is *k-distance-transitive* if, for any pairs x_1, x_2 and y_1, y_2 such that $0 \leqslant \operatorname{dist}_G(x_1, x_2) = \operatorname{dist}_G(y_1, y_2) \leqslant k$, there exists an automorphism φ of G such that $\varphi(x_i) = y_i$ for $i = 1, 2$. In view of Theorems 3.2 and 3.3 the following problem seems natural.

PROBLEM. Does there exist a fixed natural number q such that every connected, locally finite, q-distance-transitive graph with more than one end must have a cut vertex?

Perhaps this is true even for $q = 2$. If true, this would quickly imply Macpherson's characterization of the infinite, connected, locally finite graphs, that are *distance-transitive*, i.e., k-distance-transitive for all k [11]. (Reference [1] contains a short proof of the fact that such graphs must have more than one end.)

4. ENDS IN VERTEX-TRANSITIVE GRAPHS

In this section we define and review some basic properties of ends in graphs. We prove that no thin end in a vertex-transitive graph can have a 2-way infinite geodesic.

If G is a graph and P_1 and P_2 are 1-way infinite paths in G, then we write $P_1 \sim P_2$ if there are infinitely many disjoint paths in G between P_1 and P_2. Equivalently, if S is a finite vertex set of G, then G-S has a component which contains all but finitely many vertices of $P_1 \cup P_2$. Clearly, $\sim$ is an equivalence relation. An equivalence class is called an *end* of G. Ends were introduced for locally finite graphs and finitely generated groups by Freudenthal [7]. For arbitrary graphs they were independently introduced and studied extensively by Halin in [8] and subsequent papers. We say that two ends ω_1 and ω_2 in G are *G-equivalent* if there exists an automorphism of G taking ω_1 into ω_2 (or, more precisely, taking all paths of ω_1 onto all paths of ω_2). If X is a finite set of vertices and/or edges in a graph G, and ω is an end of G, then G-X has precisely one component H such that every path in ω has infinite intersection with H. We say that *ω is in H*. If G has an end ω' such that ω' is in a component H' of $G - X$

distinct from H, then we say that X *separates* ω and ω'. We also say that X separates ω from any subgraph of $G-(H \cup X)$.

Now let G be a connected, locally finite graph and let ω be an end. A *defining sequence of vertex* (*or edge*) *sets* for ω is a sequence $X_0, X_1, \ldots$, of pairwise disjoint vertex (or edge) sets with the following property: For each $i \geqslant 0$, we let G_i be the component of $G-X_i$ which contains ω. Now the condition is that, for each $i \geqslant 0$ $X_{i+1} \subseteq V(G_i) \cup E(G_i)$ and $G_{i+1} \subseteq G_i$. Then each path in ω has infinite intersection with $X_0 \cup X_1 \cup \cdots$. Indeed, if P is in ω but disjoint from $X_0 \cup X_1 \cup \cdots$, then the sequence $\operatorname{dist}_G(X_0, P)$, $\operatorname{dist}_G(X_1, P), \ldots$ is strictly decreasing, a contradiction. Conversely, if a path Q has infinite intersection with $X_0 \cup X_1 \cup \cdots$, then Q is in the same end as any fixed path P in ω because P has infinite intersection with $X_0 \cup X_1 \cup \cdots$ and each G_i is connected. We say that ω has *size m*, and we write $s(\omega)=m$, if a defining sequence of vertex sets $X_0, X_1, \ldots$ can be chosen so that they all have cardinality m, and m is minimal with this property. If such an m exists, then ω is called *thin*. Otherwise ω is *thick*. The size of an end is by Halin [8] denoted $m_1(\omega)$. We also consider the maximum distance d_i in G between vertices in X_i. If there exists a (smallest) natural number d such that a defining sequence of vertex sets $X_0, X_1, \ldots$ can be chosen so that $d_i \leqslant d$ for all i, then ω has *diameter d* and we say that ω is *slim*. Otherwise ω is *fat*. Slim ends were introduced by Picardello and Woess [13] who used them to study harmonic functions on graphs.

We shall also use the *edge-size* $\operatorname{es}(\omega)$ of an end ω, which is defined in the same way as the size, except with a defining sequence of edge sets instead of vertex sets. For each end ω in an r-regular graph we have

$$s(\omega) \leqslant es(\omega) \leqslant s(\omega)\, r.$$

Now we consider a special type of defining sequence. Let S_0 be a smallest vertex set which separates G. Let G_1 be the component of $G-S_0$ which contains ω. Let S_1 be a smallest vertex set in G_1 such that every path in ω starting in S_0 intersects S_1. (It is easy to see that such an S_1 exists.) Let G_2 be the component of G_1-S_1 which contains ω. Let S_2 be a smallest vertex set in G_2 such that every path in ω starting in S_1 intersects S_2. In this way we define sequences $G \supset G_1 \supset G_2 \supset \cdots$ and $S_0, S_1, \ldots$. Clearly $|S_0| \leqslant |S_1| \leqslant \cdots$. By Menger's theorem G contains $|S_i|$ pairwise disjoint paths from S_i to S_{i+1}. For $1 \leqslant i<j$, the $|S_i|$ paths from S_i to S_{i+1} and the $|S_j|$ paths from $|S_j|$ to $|S_{j+1}|$ are pairwise disjoint (except that they have vertices of S_{i+1} in common when $j=i+1$). So the union of all these paths is a system of 1-way infinite paths in ω. If $|S_n| \rightarrow m<\infty$ as $n \rightarrow \infty$, then $s(\omega) \leqslant m$ and ω has m pairwise disjoint paths. Clearly, ω does not have $s(\omega)+1$ pairwise disjoint 1-way infinite paths. So, $m=s(\omega)$ is the maximum number of pairwise disjoint 1-way infinite paths in ω. If

$|S_n| \to \infty$ as $n \to \infty$, then ω has infinitely many 1-way infinite disjoint paths and hence ω is thick. Conversely, if ω is thick, then $|S_n| \to \infty$ as $n \to \infty$. So every thick end has infinitely many pairwise disjoint 1-way infinite paths. By similar arguments, $\mathrm{es}(\omega)$ is the maximum number of pairwise edge-disjoint 1-way infinite paths in ω when ω is a thin end.

We shall prove that, in a vertex-transitive graph, the slim ends and the thin ends are the same. For this we extend a basic and useful observation of Dunwoody [4, 2.5].

PROPOSITION 4.1. *Let G be an infinite connected graph, let e be an edge of G, and let k be a natural number. Then G has only finitely many tight k-cuts that contain e.*

Proof. The proof is by induction on k. For $k = 1$, there is nothing to prove. So assume that $k \geq 1$. We can assume that $e = xy$ is in some tight k-cut. Hence $G - e$ has a path P from x to y. Now every tight k-cut that contains e also contains an edge of P. By the induction hypothesis, there are only finitely many tight $(k-1)$-cuts in $G - e$ containing an edge of P. ∎

Let us define a *tight k-vertex-cut* of a connected graph G as a set of k vertices such that $G - S$ is disconnected and has at least two components each of which is joined to all vertices of S. (So S is relatively minimal with respect to separating these two components.)

PROPOSITION 4.2. *Let G be a connected, locally finite graph, let v be a vertex, and let k be a natural number. Then G has only finitely many tight k-vertex-cuts containing v.*

Proof. Let H be a finite subgraph of $G - v$ with the following property: If two neighbours of v belong to the same component of $G - v$ they also belong to the same component of H. If $k \geq 2$, then every tight k-vertex-cut S in G containing v also contains a vertex of H. Also, $S \setminus \{v\}$ is a tight $(k-1)$-vertex-cut in $G - v$. Now the proof is completed by induction. ∎

COROLLARY 4.3. *Let G be a connected, locally finite, vertex-transitive graph and let k be a natural number. Then, up to G-equivalence, G has only finitely many tight k-cuts and only finitely many tight k-vertex-cuts.*

In a graph with bounded vertex degrees any slim end must be thin. The converse is not true in general. But it is for vertex-transitive graphs.

THEOREM 4.4. *In every connected, locally finite, vertex-transitive graph, every thin end is slim.*

Proof. Let ω be a thin end, and let $S_0, S_1, \ldots,$ be a defining sequence of vertex sets such that $|S_i| = q = s(\omega)$ for $i \geq 0$. Let $Q_1, Q_2, ..., Q_q$ be q pairwise disjoint 1-way infinite paths in ω starting in S_0. Let H be a finite connected subgraph containing S_0. Let m be a natural number such that the 1-way infinite subpaths of Q_i starting in S_m do not intersect H. Then the sets S_m, S_{m+1}, ... are tight q-vertex-cuts. By Corollary 4.3, there are only finitely many types of them, up to G-equivalence. Hence the diameter of ω is finite. ∎

Clearly, a slim end has no 2-way infinite geodesics. So Theorem 4.4 implies the following:

COROLLARY 4.5. *In a connected, locally finite, vertex-transitive graph no thin end has a* 2-way *infinite geodesic.*

Easy examples show that Corollary 4.5 does not generalize to non-transitive graphs. Corollary 4.5 raises the question if every thick end in a connected, locally finite, vertex-transitive graph has a 2-way infinite geodesic. We consider this problem in the next sections.

5. ENDS AND GEODESICS

In this section we prove that every connected, locally finite, vertex-transitive, inaccessible graph has a thick end with a 2-way infinite geodesic.

First we derive some basic properties of geodesics. Let G be a locally finite graph and let v be a vertex in G. Let $P_1, P_2, ...$ be an infinite sequence of paths starting in v. Assume that either infinitely many P_i are 1-way infinite or that the length of P_i tends to ∞ as $i \to \infty$. Then we define a *convergent subsequence* of $P_1, P_2, ...$ as follows: We take a subsequence such that all paths in that subsequence have the same first edge. Call the first path in this subsequence P_{i_1}. From this subsequence we take a subsequence such that all paths in it have the same second edge, etc. This results in a 1-way infinite path P and a subsequence $P_{i_1}, P_{i_2}, ...$ which we say *converges to* P and we write $\lim P_{i_m} \to P$ as $m \to \infty$. Similarly, every infinite sequence of 2-way infinite paths containing some fixed vertex has a subsequence converging to a 2-way infinite path.

PROPOSITION 5.1 (Watkins [18]). *Let G be a connected, locally finite graph.*

(a) *If ω is an end of G, then some* 1-*way infinite geodesic in G belongs to ω.*

(b) *If ω_1 and ω_2 are two distinct ends in G, then G has a* 2-*way infinite geodesic P such that one end of P is in ω_1 and the other end of P is in ω_2.*

Proof of (a). Let v be a vertex and let $P: x_1 x_2 \cdots$ be a 1-way infinite path in ω. Let P_i be a geodesic from v to x_i $(i = 1, 2, \dots)$. Let $P_{i_1}, P_{i_2}, \dots$ be a convergent subsequence of $P_1, P_2, \dots$ converging to Q, say. Then Q is a 1-way infinite geodesic in ω.

Proof of (b). Let $P: x_1 x_2 \cdots$ and $Q: y_1 y_2 \cdots$ be 1-way infinite geodesics in ω_1 and ω_2, respectively. Without loss of generality we can assume that there exists a finite vertex set S such that P and Q belong to distinct components of $G - S$. Let R be a finite path from some x_i to some y_j such that $R \cap (P \cup Q) = \{x_i, y_j\}$. Let d be the length of R. Among all such R we choose one such that $d - i - j$ is minimum. (It is easy to see that $d - i - j$ cannot be arbitrarily small.) Then $R \cup (x_i x_{i+1} \cdots) \cup (y_j y_{j+1} \cdots)$ is a geodesic. ∎

PROPOSITION 5.2. *Every infinite, connected, locally finite, almost transitive graph G has a 2-way infinite geodesic.*

Proof. As G is infinite, connected, and locally finite, G has, for each natural number k, a geodesic R_k of length $2k$. As G is almost transitive, we can assume that some fixed vertex v is the midvertex of R_k for infinitely many k. Let P_k, Q_k be the two paths in R_k starting at v. Take a subsequence $R_{i_1}, R_{i_2}, \dots,$ such that $P_{i_k} \to P$ and $Q_{i_k} \to Q$ as $k \to \infty$. Then $P \cup Q$ is a 2-way infinite geodesic. ∎

Proposition 5.2 was proved by Watkins [18] in the vertex-transitive case. We shall later use the extension to almost transitive graphs. It is easy to see that "almost transitive" cannot be replaced by "one-ended" in Proposition 5.2.

THEOREM 5.3. *Every connected, locally finite, vertex-transitive, inaccessible graph has a thick end with a 2-way infinite geodesic.*

Proof. Let v be any vertex of G. For each natural number k, G has two ends ω_k, ω'_k such that no set of k edges separates ω_k and ω'_k. By Proposition 5.1(b), G has two 1-way infinite paths P_k, Q_k such that P_k is in ω_k, Q_k is in ω'_k, $P_k \cup Q_k$ is a 2-way infinite geodesic and $P_k \cap Q_k$ is a single vertex. As G is vertex-transitive we can assume that $P_k \cap Q_k = \{v\}$ for each $k \geqslant 1$. Now consider a subsequence $P_{i_1} \cup Q_{i_1}, P_{i_2} \cup Q_{i_2}, \dots$ such that $P_{i_m} \to P$ and $Q_{i_m} \to Q$ as $m \to \infty$. Then $P \cup Q$ is a 2-way infinite geodesic. Let ω and ω' be the ends containing P and Q, respectively. We shall prove that $\omega = \omega'$.

Assume (reductio ad absurdum) that $\omega \neq \omega'$. Then there exists a finite vertex set and hence also a finite edge set whose deletion disconnects G such that ω and ω' are in distinct components. Hence there exists a k-cut F such that ω and ω' are on different sides of F. If we take k to be smallest possible, it is easy to see that F is a tight k-cut. (Otherwise F splits into smaller cuts one of which separates ω and ω'). Let A and B be the sides of F. Let m be the maximum distance in $G-F$ between endvertices of F belonging to the same side of F. Let x be a vertex of P and y a vertex of Q such that x and y are on different sides of F and such that the distance from $\{x, y\}$ to F is $>m$.

By definition of convergence, there exists a natural number n such that $n>k$ and $P_n \cup Q_n$ contains the subpath of $P \cup Q$ from x to y. As $P \cup Q$ and $P_n \cup Q_n$ are geodesics, the infinite subpaths of P_n and P (respectively Q_n and Q) starting at x (respectively y) do not intersect F. Hence F separates ω_n and ω'_n. But this contradicts the assumption on ω_n, ω'_n since $|F|=k<n$. So $\omega=\omega'$. By Corollary 4.5, ω is thick. ∎

The next result demonstrates the usefulness of 2-way infinite geodesics.

THEOREM 5.4. *Let G be a graph as in Theorem 5.3. Then G has a thick end ω with the following property: For every natural number n, there exists an end ω' such that ω and ω' cannot be separated by fewer than n vertices.*

Proof. Let v be a fixed vertex of G and let ω_k, ω'_k be two ends of G that cannot be separated by any set of k vertices. Let P_k, Q_k, P, Q, and ω be as in the proof of Theorem 5.3. Suppose (reductio ad absurdum) that n is a natural number such that ω can be separated from each other end by n vertices. In particular, for each $k \geqslant n$, there exists a (minimal) set V_k of at most n vertices such that V_k separates ω from ω_k or ω'_k. Let G_k be the component of $G-V_k$ containing ω, and let G'_k be the component of $G-V_k$ containing ω_k (or ω'_k). Then G'_k contains both ω_k and ω'_k. In particular, G'_k contains all but finitely many vertices of $P_k \cup Q_k$. The minimality of V_k implies that V_k is tight. By Corollary 4.3, there are only finitely many non-G-equivalent tight m-vertex-cuts for $m \leqslant n$. Hence there exists a natural number d (independent of k) such that any two vertices of V_k have distance $\leqslant d$ in G. As ω is in G_k, G_k contains all but finitely many vertices of $P \cup Q$. Now v is on both $P \cup Q$ and $P_k \cup Q_k$. As they are both geodesics, $\mathrm{dist}_G(v, V_k) \leqslant d$. From this it follows by Proposition 4.2 that for some infinite subsequence $n<k_1<k_2< \cdots$ we have $V_{k_1}=V_{k_2}= \cdots$. Assume without loss of generality that $V_{k_1}=V_k$. It also follows from the proof of Theorem 5.3 that every convergent subsequence of $P_{k_1} \cup Q_{k_1}$, $P_{k_2} \cup Q_{k_2}$, ... converges to a two-way infinite path in ω, i.e., to a path which has only a finite vertex set outside G_k. $P_{k_i} \cup Q_{k_i}$ is a geodesic in an end separated from ω by $V_{k_i}=V_k$. Therefore all vertices which are in both $P_{k_i} \cup Q_{k_i}$ and G_k have distance $\leqslant d$ from $V_{k_i}=V_k$. Hence $P_{i_1} \cup Q_{i_1}$, $P_{i_2} \cup Q_{i_2}$, ... has no subsequence converging to a path in ω. This contradiction completes the proof. ∎

DEFINITION. For an end ω in a graph G we denote by $k(\omega)$ the smallest number k such that ω can be separated from any other end by at most k vertices. If this number does not exist, we put $k(\omega)=\infty$.

For a thin end ω of size m we have $k(\omega)\leqslant m$. Hence Theorem 5.4 implies:

COROLLARY 5.5. *A connected, locally finite, vertex-transitive graph is accessible if and only if, for each end ω we have $k(\omega)<\infty$.*

In particular, we see that if $k(\omega)<\infty$ for every end ω, then there is a finite upper bound on the numbers $k(\omega)$.

6. ENDS AND STRUCTURE TREES OF FAMILIES OF PAIRWISE NON-CROSSING CUTS

In this section we apply the powerful theory of non-crossing cuts and structure trees developed by Dicks and Dunwoody [1]. Structure trees were introduced by Dunwoody [3] and are also applied by Möller [12].

We shall here relate the theory to the end structure of vertex-transitive graphs, in particular accessible vertex-transitive graphs.

For the sake of completeness we review basic properties of structure trees. Let us consider a connected, locally finite graph G, a natural number k, and a family $\mathscr{F}$ of non-empty pairwise non-crossing tight cuts, each of cardinality $\leqslant k$. Let $\mathscr{F}_s$ denote the family of sets A such that A is a side of a cut in $\mathscr{F}$. Note, that $\mathscr{F}_s$ contains neither $\varnothing$ nor $V(G)$. Then $\mathscr{F}_s$ is partially ordered by inclusion and has the additional properties:

(i) If $A\in\mathscr{F}_s$, then $\bar{A}=V(G)\backslash A\in\mathscr{F}_s$.

(ii) *If* $A, B\in\mathscr{F}_s$, then one of $A\cap B$, $A\cap\bar{B}$, $\bar{A}\cap B$, $\bar{A}\cap\bar{B}$ is empty.

(iii) If $A\in\mathscr{F}_s$, $B\in\mathscr{F}_s$, $A\subseteq B$, then there are only finitely many $A'\in\mathscr{F}_s$ such that $A\subset A'\subset B$.

(iv) If $A_1, A_2, \ldots\in\mathscr{F}_s$ and $A_1\supset A_2\supset\cdots$, then $\bigcap_{i=1}^{\infty} A_i=\varnothing$.

Part (iii) follows from Proposition 4.1 applied to the edges in a path from A to $\bar{B}$. Proposition 4.1 also implies (iv).

Now form a new graph $T=T(\mathscr{F})$ as follows: First we form a 1-regular graph H such that every edge e in H corresponds to a cut F in $\mathscr{F}$. The endvertices of e correspond to the two sides of F. Consider two edges e_1, e_2 corresponding to cuts F_1, F_2. Let A_i, B_i be the sides of F_i, and let x_i, y_i be the endvertices of e_i corresponding to A_i, B_i for $i=1,2$. We write $y_1\sim x_2$ if $A_1\subseteq A_2$ and there is no A in $\mathscr{F}_s$ such that $A_1\subset A\subset A_2$ properly.

It is easy to show that $\sim$ is an equivalence relation. Now T is obtained by identifying all vertices in an equivalence class into a single vertex. It is a straightforward exercise to show that T is a tree, which we call the *structure tree* for $\mathscr{F}$. A side in G of a cut F in $\mathscr{F}$ corresponds to an endvertex and hence to a side in T of the edge e in T corresponding to F. Hence there is a natural map $g: V(G) \to V(T)$ with the following property: If u is a vertex of G, $F \in \mathscr{F}$, and e is the edge of T corresponding to F, then the side of F containing u corresponds to the side of e in T containing $g(u)$. (For fixed u in $V(G)$, there is only one vertex $g(u)$ in $V(T)$ with this property.) Note that g need not be 1–1 if $\mathscr{F}$ is sparse. Also, g need not be onto. For example, if all degrees of G are $\leqslant k$ and $\mathscr{F}$ is the family of cuts each consisting of all edges incident with a single vertex, then g maps $V(G)$ onto $V(T)$ minus just one vertex v. That vertex v is joined to all other vertices of T. This example also shows that T need not be locally finite. Of course, T is always countable.

Suppose now that P is a 1-way infinite path in G. Then we form an infinite walk W_P (i.e., a sequence of neighbouring vertices where repetition is allowed, even for successive vertices) as follows: If $P = u_1 u_2 \cdots$, then we let $v_i = g(u_i)$ be the vertex in T corresponding to u_i. We let W_P be the path from v_1 to v_2 followed by the path from v_2 to v_3 followed by the path from v_3 to v_4, etc. The edges in the path from v_i to v_{i+1} correspond to the cuts in $\mathscr{F}$ containing the edge $u_i u_{i+1}$. Thus W_P is a 1-way infinite walk in T. W_P uses no edge of T more than k times. Consider now the following two cases for W_p.

Case 1. No vertex of T is visited infinitely often by W_P.

Then we can inductively transform W_P into a 1-way infinite path W'_P of T as follows: When W'_P uses a vertex v, then the next edge in W'_P is the last edge incident with v used by W_P. Thus W'_P defines an end of T. If Q is another path in G which is in the same end ω as P, then it is not hard to see that W'_Q defines the same end of T. Hence we may write $g(\omega)$ for the latter and say that it *corresponds* to ω.

In this way, every end of T is of the form $g(\omega)$ for precisely one end ω of G, and ω has edge-size $\leqslant k$. (Note that the uniqueness of ω follows from Proposition 4.1: The edge sequence of any 1-way infinite path in T has a subsequence which corresponds to a defining sequence of edge sets in G.)

Case 2. There is a vertex v of T visited infinitely often by W_P.

Then this vertex is unique. Again, if Q is another path in G which is in the same end ω as P, then W_Q visits the same vertex v infinitely often. We write $g(\omega) = v$ and say that ω *lives* in v. (Our example following the definition of a structure tree shows that a vertex of T may be the image under g of an end of G, but not a vertex.)

Thus, g is a mapping from vertices and ends of G to vertices and ends of T. While the preimage of an end of T is a unique end of G, there may be vertices of T whose preimages contain more than one vertex and/or end of G, or none.

A different approach for describing the correspondence between ends of G and ends and vertices of T was used by Woess [20] (where T is not used explicitely) and Möller [12].

Now suppose, in addition, that G is vertex-transitive and that $\mathscr{F}$ is invariant under Aut(G). That is, if $F \in \mathscr{F}$ and $\varphi \in \text{Aut}(G)$, then $\varphi(F) \in \mathscr{F}$. Then each automorphism φ of G induces an automorphism of T. By Corollary 4.3, $\mathscr{F}$ has only finitely many types of non-G-equivalent cuts. (Recall that we assume that $|F| \leqslant k$ for each $F \in \mathscr{F}$.) Hence $E(T)$ (and then also $V(T)$) has only finitely many orbits under the group of automorphisms of T induced by Aut(G).

Now consider the Boolean ring $B(G)$ consisting of all sides of finite cuts in G (including $V(G)$ and the empty set). Multiplication is intersection and addition is symmetric difference. Let $B_n(G)$ be the subring generated by the sides of k-cuts, $k \leqslant n$. The following powerful result is due to Dicks and Dunwoody [1, II 2.20].

THEOREM 6.1. *Let G be a connected, locally finite graph and let n be a natural number. Then G contains a collection $\mathscr{F}_n$ of non-empty, pairwise non-crossing, tight cuts of cardinality $\leqslant n$ such that $\mathscr{F}_n$ is invariant under* Aut(G), *and $B_n(G)$ is generated (as a ring) by the sides of cuts in $\mathscr{F}_n$.*

We may also consider $B(G)$ as an Aut(G)-module, i.e., we ignore the multiplication. (For the precise definition, see [1].) Dicks and Dunwoody [1, IV 7.6] proved that a finitely generated group is accessible if and only if, for each of its connected, locally finite Cayley graphs G, $B(G)$ (considered as an Aut(G)-module) is finitely generated. This is equivalent with the following (when restricted to Cayley graphs).

PROPOSITION 6.2. *Let G be a connected, locally finite, vertex-transitive graph. Then $B(G)$ is finitely generated as an* Aut(G)-*module if and only if there exists a natural number n such that $B(G) = B_n(G)$.*

Proof. If $B(G)$ is fintely generated as an Aut(G)-module, then we let n be the largest size of the cuts of which the generators are sides. Then clearly $B_n(G) = B(G)$. Conversely, let us assume that $B_n(G) = B(G)$. Let $\mathscr{F}_n$ be as in Theorem 6.1. There are only finitely many non-G-equivalent cuts in $\mathscr{F}_n$, by Proposition 4.1. These generate $B(G)$ as an Aut(G)-module. Note that when we speak of a module we ignore the multiplication in $B(G)$ (i.e., intersection of sides). But it is easy to see that one does not need multiplication. For, if A and B are sides of non-crossing cuts, then either $A \cap B$

is one of A, B, $\emptyset$, or else $A \cap B = V(G) \backslash [(A \backslash B) \cup (B \backslash A)]$ which shows that $A \cap B$ can be obtained from sides of cuts using symmetric difference only. ∎

It follows from the results of the next section that the graphs satisfying the statements in Proposition 6.2 are precisely the accessible vertex-transitive graphs.

7. The End Structure of Accessible, Vertex-Transitive Graphs

In this section we give a precise description of the end structure of the connected, locally finite, accessible, vertex-transitive graphs.

First we return to the structure tree associated with the collection $\mathscr{F}_n$ of cuts in Theorem 6.1. The following was observed by Möller [12].

PROPOSITION 7.1. *Let G be a connected, locally finite graph and n a natural number. Let $\mathscr{F}_n$ be as in Theorem 6.1. Let F be a cut with sides A, $B \in B_n(G)$. Then any two ends ω_1, ω_2 in $G(A)$ and $G(B)$, respectively, can be separated by a cut in $\mathscr{F}_n$.*

Proof. There exist cuts $F_1, F_2, ..., F_q$ in $\mathscr{F}_n$ with sides A_i and B_i ($i = 1, 2, ..., q$) such A is the symmetric difference of $A_1, A_2, ..., A_q$. For $i = 1, 2$, let P_i be a 1-way infinite path in ω_i such that P_i is either contained in or disjoint from each A, $A_1, A_2, ..., A_q$. For some $j = 1, 2, ..., q$ we have $V(P_1) \subseteq A_j$ and $V(P_2) \subseteq B_j$ or vice versa. For otherwise, $V(P_1)$ and $V(P_2)$ would both be contained in or both be disjoint from the symmetric difference of $A_1, A_2, ..., A_q$, a contradiction. ∎

COROLLARY 7.2. *Let G and $\mathscr{F}_n$ be as in Proposition 7.1. Assume, in addition, that $B_n(G) = B(G)$. Let T_n be the structure tree associated with $\mathscr{F}_n$. If ω_1 and ω_2 are two distinct ends of G, then T_n has an edge e which separates $g(\omega_1)$ and $g(\omega_2)$ in T_n. In particular, for each vertex v of T_n, at most one end of G lives in v.*

We consider again the graph G and the collection $\mathscr{F}_n$ in Theorem 6.1. Assume, in addition, that G is vertex-transitive. Let $T_n = T(\mathscr{F}_n)$ be the structure tree associated with $\mathscr{F}_n$ and let v be any fixed vertex in T. Let e be any edge incident with v and let F be the corresponding cut in G. Let $A(v, e)$ be the side of F such that the corresponding side of e in T contains v. For any natural number q we let $R_q(v, e)$ denote the subgraph in $G - A(v, e)$ induced by the vertices of distance at most q from F. We choose q so large that $R_q(v, e)$ contains all geodesics in $G - A(v, e)$ between endvertices of F and so that also the following is satisfied: If $P_1, P_2, ..., P_r$

are pairwise edge-disjoint paths in $G - A(v, e)$ such that the endvertices of each P_i $(1 \leqslant i \leqslant r)$ are endvertices of F, then $R_q(v, e)$ contains pairwise edge-disjoint paths $P'_1, P'_2, ..., P'_r$ such that P'_i and P_i have the same endvertices for $i = 1, 2, ..., r$. As there are only finitely many non-G-equivalent cuts in $\mathscr{F}_n$ we can choose the same q for all edges in T_n. For this q we put $R_q(v, e) = R(v, e)$. We also let $A(v)$ be the intersection of all $A(v, e)$ taken over all edges of T incident with v. (Then $A(v) = \{u \in V(G) \mid g(u) = v\}$.) Now let G_v be the union of $G(A(v))$, all cuts F corresponding to edges in T incident with v, and all graphs $R(v, e)$ such that e is incident with v. It is easy to see that the following hold.

(i) G_v is a connected graph. (A geodesic in G between two vertices x, y in G_v can easily be transformed into a path in G_v between x and y.)

(ii) If P is a 1-way infinite geodesic in an end ω of G, then ω lives in v if and only if an infinite subpath of P is in G_v.

(iii) G_v is almost transitive. (As G is vertex-transitive, $E(T)$ has only finitely many orbits under the group of automorphisms induced by $\mathrm{Aut}(G)$.)

Now, if $d_T(v) < \infty$, then G_v is finite because every vertex in $A(v)$ is incident with a cut F in $\mathscr{F}_n$ (by the vertex-transitivity of G). In that case no end of G can live in V. Consider next the case $d_T(v) = \infty$. By Proposition 5.2, G_v has a 2-way infinite geodesic P which is also a geodesic of G. Both ends of P live in v. Combining these observations with Corollaries 7.2 and 4.5 we obtain:

THEOREM 7.3. *Let G be a connected, locally finite, vertex-transitive graph and assume that n is a natural number such that $B_n(G) = B(G)$. Let $\mathscr{F}_n$ be as in Theorem* 6.1 *and let $T = T_n$ be its structure tree. If v is a vertex of v and $d_T(v) < \infty$, then G_v is finite and no end of G lives in v. If $d_T(v) = \infty$, then G_v is infinite, almost-transitive, and precisely one end of G lives in v. That end has a 2-way infinite geodesic and is therefore thick. Every thick end of G lives in some such v, and the thin ends of G correspond to the ends of T_n.*

COROLLARY 7.4. *Let G be a connected, locally finite, vertex-transitive graph. Suppose $B_n(G) = B(G)$ for some n and let G, n, $\mathscr{F}_n$, and T_n be as in Theorem* 7.3. *Then any two ends of G can be separated by n edges, and every thin end has edge-size $\leqslant n$.*

Proof. By Theorem 7.1, any two ends of G can be separated by a cut in $\mathscr{F}_n$. As a thin end in G corresponds to an end in T_n, it has at most n pairwise edge-disjoint paths. ∎

By Corollary 7.4, the graphs in Proposition 6.2 are accessible.

COROLLARY 7.5. *Let G be a connected, locally finite, vertex-transitive graph. Suppose $B_n(G) = B(G)$ for some n. Then G has only countably many thick ends and only finitely many types of non-G-equivalent thick ends.*

Proof. T_n has only countably many edges as every edge of T_n corresponds to a finite edge set of G. So T_n has only countably many vertices, and every thick end of G lives in some vertex of T_n. Moreover, distinct thick ends of G live in distinct vertices of T_n. The last statement in Corollary 7.5 follows since T_n is almost transitive. ∎

Woess [19] showed that each connected, locally-finite, vertex-transitive graph G with no thick ends is "tree-like." This also follows from Theorems 5.3 and 7.3 combined with Proposition 7.6 below, since each G_v is finite. This means that G "looks like" T_n. Using this, it is easy to show that G is also tree-like both in the sense of Woess [19] and in the sense of Diestel [2].

THEOREM 7.6. *A connected, locally-finite, vertex-transitive graph G is accessible if and only if $B_n(G) = B(G)$ for some natural number n.*

Proof. After Corollary 7.4 we observed that the "if part" holds. So let G be a connected, locally-finite, vertex-transitive graph, and let n be a natural number such that any two ends can be separated by a k-cut, $k \leqslant n$. Assume that n is at least the vertex degree of G. We shall show that $B_n(G) = B(G)$. Let $\mathscr{F}_n$ be as in Theorem 6.1, and let T_n be its structure tree. For any two ends ω_1, ω_2 in G, there is a k-cut F ($k \leqslant n$) such that ω_1 and ω_2 are in distinct sides A_1 and A_2 of F. As $A_1 \in B_n(G)$, there is some edge e in T_n such that the cut in G corresponding to e separates ω_1 and ω_2, by Proposition 7.1.

Now let $A \in B(G)$. We shall prove that $A \in B_n(G)$. By the preceding remark, if ω_1 in an end of G in $G(A)$ and ω_2 is an end of G in $G - A$, then some edge of T_n separates the corresponding ends or vertices in T_n. We claim that T_n has a finite edge set E_n such that, if ω and ω' are ends in $G(A)$ and $G - A$ respectively, then the corresponding ends in T_n belong to distinct components of $T_n - E_n$. Assume this claim were false. Put $E(T_n) = \{e_1, e_2, ...\}$. Let ω_k and ω'_k be ends in $G(A)$ and $G - A$, respectively, such that the ends in T_n corresponding to ω_k, ω'_k belong to the same component of $T_n - \{e_1, e_2, ..., e_k\}$. Let P_k be a 2-way infinite geodesic in G representing ω_k and ω'_k. As P_k intersects the finite cut with sides A and $V(G) \setminus A$ there is a subsequence of $P_1, P_2, ...$ which converges to, say P. Then P is a 2-way infinite geodesic whose ends are in $G(A)$ and $G - A$, respectively. But now it is easy to see that the two ends of P cannot be separated by a cut in $\mathscr{F}_n$.

This contradiction proves the claim that, for any ends ω, ω' in $G(A)$ and $G-A$, respectively, the corresponding ends in T are in distinct components of $T-E_n$. The set of vertices in those components of $T-E_n$ corresponding to ends in $G(A)$ is denoted A'. Let F' be the set of edges in T from A' to $V(T)\setminus A'$. Now F' corresponds to a cut F in G. (Two vertices u and v in G are separated by F iff the corresponding vertices $g(u)$ and $g(v)$ in T_n are separated by F'.) Let B and $V(G)\setminus B$ be the sides of F. Not both sets $A \cap B$ and $B\setminus A$ are infinite since otherwise $G(A \cap B)$ and $G(B\setminus A)$ would have ends which would be in $G(B)$, but in distinct graphs $G(A)$, $G-A$ which contradicts the way B is defined. Similarly, one of $A\setminus B$ and $(V(G)\setminus A)\setminus B$ is finite. But then A differs from B or $V(G)\setminus B$ or $V(G)$ or $\varnothing$ by a finite vertex set. Since $B_n(G)$ contains B and every finite vertex set of G (because it contains every singleton), $A \in B_n(G)$. ∎

Corollary 4.5, Theorem 7.3, and Theorem 7.6 show that an end ω in a connected, locally-finite, accessible, vertex-transitive graph has a 2-way infinite geodesic if and only if it is thick. Also, if v denotes the vertex in T_n in which ω lives (where n is such that $B_n(G)=B(G)$), then all 2-way infinite geodesics in ω are in G_v. We have previously observed that G_v is connected and almost transitive. Furthermore, under the above assumptions we have:

Proposition 7.7. *G_v has only one end.*

Proof. Suppose P is a 2-way infinite geodesic in G_v (and hence also a geodesic in G) such that the two ends of P are separated by a finite cut F_0 in G_v. If F is a cut corresponding to an edge e in T_n incident with v, then we delete F from G whenever $(F \cup R(v, e)) \cap F_0 \neq \varnothing$. We also delete F_0 and call the resulting graph H. As we have deleted only finitely many edges of G, P minus the deleted edges has two 1-way infinite paths P_1 and P_2. Now H has no path between P_1 and P_2. Hence P_1 and P_2 belong to distinct ends of G both living in v. This contradiction to Corollary 7.2 shows that G_v has only one end, as claimed. ∎

Now let G_v' denote the union of all 2-way infinite geodesics in G_v (that is, the 2-way infinite geodesics representing the end ω living in v). Clearly, all 1-way infinite paths in G_v' belong to ω (as they live in v). But we do not know whether G_v' is connected or almost transitive or one-ended like G_v.

8. Ends in Inaccessible Vertex-transitive Graphs

We have already seen (Theorem 5.4) that a connected, locally finite, vertex-transitive, inaccessible graph must have a thick end which behaves badly in the sense that it cannot be separated from every other end by few edges. In this section we show that also the thin ends in such a graph behave in an uncontrolled way in the sense that there is no upper bound on their size.

Let G, n and $\mathscr{F}_n$ be as in Theorem 6.1. Let T_n be the structure tree associated with $\mathscr{F}_n$ and assume that G is vertex-transitive. Now we form, for each vertex v in T_n, the almost transitive graph G_v as after Corollary 7.2. (As we are not assuming now that $B_n(G) = B(G)$, Corollary 7.2 no longer holds.)

If an end ω of G lives in v, then it can be represented by a 1-way infinite path in G_v and hence it defines an end in G_v which we also denote by ω. Moreover, if ω has q edge-disjoint 1-way infinite paths in G, then it follows from the construction of G_v, that ω also has q edge-disjoint paths in G_v. Hence we have:

LEMMA 8.1. *An end ω in G living in G_v has the same edge-size in G as in G_v. In particular, ω is thin in G_v if and only if it is thin in G.*

LEMMA 8.2. *A thin end ω of G lives in a vertex of T_n if and only if ω has edge-size $> n$.*

Proof. As observed in Section 6, every 1-way infinite path in T_n corresponds to an end in G of edge-size $\leqslant n$. Suppose conversely that ω is a thin end in G of edge-size $\leqslant n$. Let $F_0, F_1, ...,$ be a defining sequence of edge sets for ω, each of size $\mathrm{es}(\omega)$. Let $G_0 \supseteq G_1 \supseteq \cdots$ be the sequence of subgaphs of G such that G_{i+1} is the component of $G - F_i$ containing ω for $i = 0, 1, ...$. Let P be a path in ω chosen so that P has precisely one edge in common with F_i for $i = 0, 1, ...$. Now $V(G_{i+1})$ is the symmetric difference of sides of cuts in $\mathscr{F}_n$. Hence F_i is the symmetric difference of cuts in $\mathscr{F}_n$, ($i = 0, 1, ...$). One of these, say F_i', has odd intersection with $F_i \cap E(P)$ (because F_i has odd intersection with $E(P)$), $i = 0, 1, ...$. Now the walk W_P in T_n corresponding to the path P in G uses the edge e_i in T_n corresponding to F_i' an odd number of times $i = 0, 1, ...$. Clearly, the set $\{e_0, e_1, ...\}$ is infinite. It follows that ω does not live in any vertex of T_n because that would imply that W_P uses only finitely many edges of T_n an odd number of times. ∎

The following was proved by Jung [10]. (Jung's result was stated for vertex-transitive graphs but the proof easily generalizes to almost transitive graphs.)

LEMMA 8.3. *If a connected, locally finite, almost transitive graph has more than one end, then it has a thin end.*

Now assume that G is connected, locally-finite and vertex-transitive and that G has two ends ω_1 and ω_2 which cannot be separated by any k-cut where $k \leqslant n$. Since the corresponding ends (or vertices) in T_n cannot be separated by an edge of T_n, ω_1, and ω_2 must live in the same vertex v of T_n. Any 2-way infinite geodesic whose ends are in ω_1 and ω_2, respectively, is in G_v. Since ω_1 and ω_2 can be separated by a finite cut in G, they can also be separated by a finite cut in G_v. By Lemma 8.3, G_v has a thin end ω. By Lemma 8.1, ω is thin in G, too. By Lemma 8.2, $\mathrm{es}(\omega) \geqslant n+1$. Hence we obtain

THEOREM 8.4. *If G is a connected, locally finite, vertex-transitive, inaccessible graph, then, for each natural number n, G has a thin end of size $\geqslant n$.* ∎

Combining this with Theorem 7.3 we obtain the following.

COROLLARY 8.5. *A connected, locally-finite, vertex-transitive graph G is accessible if and only if there is a natural number n such that each thin end of G has size $\leqslant n$.*

In Corollary 8.5 we may replace "thin" by "slim." This answers [19, Problem 2] affirmatively. Thus [19, Problem 1] is another version of the problem of accessibility.

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Note added in proof. R. Möller has settled the problem at the end of Section 3 affirmatively for $q = 2$.

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